Math 217 Fall 2025

Quiz 11 – Solutions

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1. Complete* the partial sentences below into precise definitions for, or precise mathematical characterizations of, the italicized term:

   (a) Suppose V and W are vector spaces. A *linear transformation* $T : V \to W$ is ...

   Solution: A function $T : V \to W$ such that for all $u, v \in V$ and all scalars α (over the underlying field),

   $$T(u + v) = T(u) + T(v) \quad \text{and} \quad T(\alpha v) = \alpha\, T(v).$$

   Equivalently, $T(\alpha u + \beta v) = \alpha T(u) + \beta T(v)$ for all scalars α, β and all $u, v \in V$.

   (b) To say that a list of vectors $(x_1, x_2, \ldots, x_d)$ in a vector space X is *linearly independent* means ...

   Solution: That the only scalars $a_1, \ldots, a_d$ satisfying

   $$a_1x_1 + \cdots + a_dx_d = 0_X$$

   are $a_1 = \cdots = a_d = 0$. Equivalently, no x_j can be expressed as a linear combination of the others.

   (c) Suppose U is a vector space and $u_1, \ldots, u_n \in U$. The *span* of $(u_1, \ldots, u_n)$ is ...

   Solution: The set of all finite linear combinations of the vectors:

   $$\operatorname{span}\{u_1, \ldots, u_n\} = \left\{ \sum_{i=1}^{n} \alpha_i u_i \;\middle|\; \alpha_1, \ldots, \alpha_n \in \mathbb{F} \right\},$$

   where $\mathbb{F}$ is the underlying field. It is the smallest subspace of U containing all u_i.

2. Suppose the list $(\vec{v}_1, \vec{v}_2)$ of vectors in a vector space V is linearly independent. Show that $\vec{v}_1$ is not a scalar multiple of $\vec{v}_2$ and $\vec{v}_2$ is not a scalar multiple of $\vec{v}_1$.

   Solution: Assume that $(\vec{v}_1, \vec{v}_2)$ is linearly independent. Assume to the contrary that $\vec{v}_1$ is a scalar multiple of $\vec{v}_2$, say $\vec{v}_1 = \lambda\vec{v}_2$ for some scalar λ. Then

   $$\vec{v}_1 - \lambda\vec{v}_2 = 0 \quad \Longrightarrow \quad 1 \cdot \vec{v}_1 + (-\lambda) \cdot \vec{v}_2 = 0.$$

*For full credit, please write out fully what you mean instead of using shorthand phrases.

Since $(\vec{v}_1, \vec{v}_2)$ is linearly independent, the only linear relation is the trivial one, so the coefficients must be $1 = 0$ and $-\lambda = 0$, a contradiction. Hence $\vec{v}_1$ is not a scalar multiple of $\vec{v}_2$. The same argument with the roles reversed shows $\vec{v}_2$ is not a scalar multiple of $\vec{v}_1$.

3. True or False. If you answer true, then state TRUE. If you answer false, then state FALSE. Justify your answer with either a short proof or an explicit counterexample.

(a) Suppose V and W are vector spaces, $T : V \to W$ is a linear transformation, and $\vec{0}_V$ is the zero vector in V. If $\ker(T) = \{\vec{0}_V\}$, then T is bijective.

Solution: FALSE. $\ker(T) = \{0\}$ means T is injective, but not necessarily surjective. Counterexample: Let $T : \mathbb{R} \to \mathbb{R}^2$ be $T(t) = (t, 0)^T$. Then $\ker(T) = \{0\}$, so T is injective, but $\operatorname{Im}(T) = \{(t, 0) : t \in \mathbb{R}\}$ is a proper subspace of $\mathbb{R}^2$, so T is not surjective and hence not bijective.

(b) The list $\left(\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \right)$ of vectors in $\mathbb{R}^3$ is linearly independent.

Solution: TRUE. Suppose

$$a \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} + b \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + c \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}.$$

Comparing coordinates gives

$$\begin{cases} a + b + c = 0 & \text{(first)} \\ a + c = 0 & \text{(second)} \\ c = 0 & \text{(third)} \end{cases} \implies c = 0,\ a = 0,\ b = 0.$$

Hence the only solution is $a = b = c = 0$, so the vectors are linearly independent.